

A New Dimensionally Reduced Effective Action for QCD at High Temperature

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Abstract: New terms are derived for the three-dimensional effective action of the static modes of pure gauge $SU(N)$ at high temperature. In previous works, effective vertices have been obtained by evaluating diagrams involving 2, 3 or 4 external static gluons with one internal nonstatic loop. I take a somewhat different approach by making a covariant derivative expansion of the one loop effective action for the static modes, keeping all terms involving up to six covariant derivatives. The resulting effective action is manifestly invariant under spatially dependent gauge transformations and contains new 5- and 6-point effective vertices.

1 Introduction

Several years ago it was realized that for sufficiently high temperatures, the dominant infrared behavior of four-dimensional QCD could be described by a static three-dimensional theory [1]. The basic idea is the following: In the imaginary time formalism of finite temperature gauge field theory, the gauge (and ghost) fields must be periodic in imaginary time τ with period $1/T$ [2]. These fields ($A_\mu^a(x)$) can thus be expanded into fourier modes $A_{\mu,n}^a(\mathbf{x}) \exp(i2n\pi T\tau)$, each of which has a propagator of the form $[\mathbf{p}^2 + (2n\pi T)^2]^{-1}$. For any nonzero n , $2n\pi T$ acts as a mass which causes the n th mode to become infinitely “heavy” as the temperature goes to infinity. Since infinitely heavy particles decouple from zero temperature field theories [3], it has been argued that the same should happen to nonstatic $n \neq 0$ modes in high temperature field theories [1]. This would imply that at sufficiently high temperatures, four di-

mensional pure gauge $SU(N)$ would reduce to three dimensional $SU(N)$ coupled to an adjoint scalar (A_0^a):

$$\int d^4x (F_{\mu\nu}^a(x))^2 \rightarrow (1/T) \int d^3x [(F_{ij}^a(\mathbf{x}))^2 + 2(D_i^{ab} A_0^b(\mathbf{x}))^2] \quad (1)$$

Unlike zero temperature field theories with heavy particles, however, this kind of complete decoupling does not in fact occur for QCD even at infinite temperature [4]. Rather, the nonstatic modes make themselves felt within loops by generating an infinite number of effective vertices for the static modes which cannot in general be ignored. Consequently, many authors have extended the above 3-dimensional theory by including the leading corrections to 2-, 3- and 4-point static vertices which come from diagrams involving one internal nonstatic loop [4, 5, 6]. Encouragingly, lattice results indicate that 3-dimensional QCD extended in this way reproduces the full 4-dimensional QCD remarkably well for temperatures above 2.5 times that of the phase transition [7]. For lower temperatures, however, the nonstatic modes contribute more significantly, thus causing the simple actions proposed so far to become inadequate to describe the full theory. It would be interesting to see if better agreement could be generated at lower temperatures by including more of the induced static vertices in the effective theory. I take a first step towards answering that question by deriving an effective action which contains new 5- and 6-point static vertices, as well as derivative corrections to known 2-, 3- and 4-point vertices. These derivative corrections essentially constitute new terms in the ($|\mathbf{p}|/T$) infrared expansion proposed in [6], where \mathbf{p} represents the external momentum of a given effective vertex.

I begin this work by making a covariant derivative expansion of the one loop effective action of $SU(N)$ in the presence of a general background field, keeping all terms with up to six covariant derivatives. I then specialize this general result (eqn. 8) to the problem at hand by considering only static background fields and restricting the one loop integration to nonstatic fluctuations. The resulting 3-dimensional effective action is manifestly invariant under any spatially-dependent gauge transformation.

2 Covariant Derivative Expansion

The one loop contribution to effective action of pure gauge SU(N) can be calculated by evaluating the following functional determinants [8, 9]:

$$S_{\text{eff}}^{(1)} = \ln\left[\frac{\det(-D^2)}{\det(-\partial^2)}\right] - \frac{1}{2} \ln\left[\frac{\det(-D^2\delta_{\mu\nu} - 2[D_\mu, D_\nu])}{\det(-\partial^2\delta_{\mu\nu})}\right], \quad (2)$$

where $D_\mu^{ab} = \partial_\mu\delta^{ab} - gf^{abc}A_\mu^c(\mathbf{x})$. To evaluate these determinants, we use the identity:

$$\ln\left[\frac{\det(K)}{\det(K_0)}\right] = -\int_0^\infty \frac{dt}{t} [\text{Tr}(e^{-tK} - e^{-tK_0})] \quad (3)$$

for operators K and K_0 .

Using plane waves to take the functional trace of the ghost operator, we arrive at the following expression [8]:

$$\ln \det\left(\frac{-D^2}{-\partial^2}\right) = -\int_0^\infty \frac{dt}{t} \int \frac{d^4x d^3p}{(2\pi)^3} T \sum_n e^{-p^2 t} \text{tr}\{\exp[(D^2 + 2iD_\alpha p_\alpha)t] \mathbf{1} - 1\}, \quad (4)$$

where the remaining trace tr is taken only over the color indices in the adjoint representation. At finite temperature, the x_0 integral in d^4x is from 0 to $1/T$, and instead of integrating over p_0 , one must sum over the modes $p_0 = 2n\pi T$. A $\mathbf{1}$ has been included at the end of the equation to emphasize the fact that the exponential operator acts on unity; so that, for example, any term in the expansion of the exponent with a ∂_α all the way to the right will vanish.

With the help of the appendix, we expand the exponent, keeping all terms involving only D_0 or up to six covariant derivatives of any kind. After integrating over d^3p , we obtain:

$$\begin{aligned} \ln \det\left(\frac{-D^2}{-\partial^2}\right) &= \int d^4x V_{\text{eff}} - \int_0^\infty \frac{dt}{t} \int \frac{d^4x}{(4\pi t)^{3/2}} \left(\frac{T}{180}\right) \sum_n e^{-p_0^2 t} \\ &\times \text{tr} \{ 15t^2[D_\mu, D_\nu]^2 + 30t^2(1 - 2p_0^2 t)([D_i, D_0]^2 + \{[D_i, [D_i, D_0]], D_0\}) \\ &+ t^3(2\mathcal{O}_1 - 3\mathcal{O}_2) + 3t^3(1 - 2p_0^2 t)[2\mathcal{P}_1 - 2(\mathcal{P}_4 + \mathcal{P}_{13}) - \mathcal{P}_6] \\ &+ 3t^3(1 - 2p_0^2 t)[-4(\mathcal{Q}_1 + \mathcal{Q}_9) - 2(\mathcal{Q}_3 + 2\mathcal{Q}_9) + 5(\mathcal{Q}_7 + 2\mathcal{Q}_{12}) + (\mathcal{Q}_8 + \mathcal{Q}_{14})] \\ &+ 2t^3(3 - 12p_0^2 t + 4p_0^4 t^2)[- \mathcal{P}_{13} - 6\mathcal{Q}_9 + 15\mathcal{Q}_{12} + 5\mathcal{Q}_{13} + 2\mathcal{Q}_{14}] \} \mathbf{1}, \quad (5) \end{aligned}$$

where the sixth order \mathcal{O} , \mathcal{P} , and \mathcal{Q} operators are defined in the appendix, and the effective potential is given by:

$$V_{\text{eff}} = - \int_0^\infty \frac{dt}{t} \frac{T}{(4\pi t)^{3/2}} \sum_n \text{tr} \{ \exp[(D_0 + ip_0)^2 t] - e^{-p_0^2 t} \} \mathbf{1} . \quad (6)$$

The gauge field determinant can be calculated using techniques similar to those used for the ghost determinant. One obtains:

$$\begin{aligned} \ln \left[\frac{\det(-D^2 \delta_{\mu\nu} - 2[D_\mu, D_\nu])}{\det(-\partial^2 \delta_{\mu\nu})} \right] &= 4 \ln \det \left(\frac{-D^2}{-\partial^2} \right) \\ &- \int_0^\infty \frac{dt}{t} \int \frac{d^4 x}{(4\pi t)^{3/2}} \frac{T}{3} \sum_n e^{-p_0^2 t} \text{tr} \{ -6t^2 [D_\mu, D_\nu]^2 + 2t^3 \mathcal{O}_2 \\ &+ t^3 (1 - 2p_0^2 t) [-4\mathcal{P}_1 + 2(\mathcal{P}_4 + \mathcal{P}_{13}) + 4(\mathcal{Q}_3 + 2\mathcal{Q}_9) - 6(\mathcal{Q}_7 + 2\mathcal{Q}_{12})] \} \mathbf{1} . \end{aligned} \quad (7)$$

Note that the factor of 4 in the first line arises simply from the trace over $\delta_{\mu\nu}$.

For convenience, we now switch to matrix notation, defining $(I_c)^{ab} = -if^{abc}$, $A_0 = I_c A_0^c$, $F_{\mu\nu} = I_c F_{\mu\nu}^c$, $(D_\mu F_{\nu\rho}) = I_c D_\mu^{cd} F_{\nu\rho}^d$, etc. The one loop contribution to the effective action then takes the following form:

$$\begin{aligned} S_{\text{eff}}^{(1)} &= - \int d^4 x V_{\text{eff}} + g^2 \int \frac{d^4 x}{(4\pi)^{3/2}} T \sum_n \int_0^\infty \frac{dt}{\sqrt{t}} e^{-p_0^2 t} \text{tr} \{ \frac{11}{12} F_{\mu\nu}^2 \\ &- \frac{1}{6} (1 - 2p_0^2 t) [F_{\mu 0}^2 + 2A_0 (D_\mu F_{\mu 0})] + \frac{1}{90} t i g F_{\mu\nu} F_{\nu\rho} F_{\rho\mu} - \frac{19}{60} t (D_\mu F_{\mu\nu})^2 \\ &+ t (1 - 2p_0^2 t) [-\frac{19}{30} i g F_{0\mu} F_{\mu\nu} F_{\nu 0} - \frac{3}{10} (D_0 F_{0\nu}) (D_\mu F_{\mu\nu}) + \frac{1}{60} (D_\mu F_{\mu 0})^2 \\ &- \frac{1}{15} i g A_0 (D_\mu F_{\mu\nu}) F_{0\nu} + \frac{19}{30} i g A_0 (D_0 F_{\mu\nu}) F_{\mu\nu} \\ &- \frac{11}{12} g^2 A_0^2 F_{\mu\nu}^2 - \frac{1}{30} A_0 (D_\mu^2 D_\nu F_{\nu 0})] \\ &+ t (3 - 12p_0^2 t + 4p_0^4 t^2) [\frac{1}{90} (D_0 F_{0\mu})^2 - \frac{1}{15} i g A_0 (D_0 F_{0\mu}) F_{0\mu} \\ &+ \frac{1}{6} g^2 A_0^2 F_{\mu 0}^2 + \frac{1}{9} g^2 A_0^3 (D_\mu F_{\mu 0}) - \frac{2}{45} A_0 (D_0^2 D_\mu F_{\mu 0})] \} \end{aligned} \quad (8)$$

Only the terms at the end of the first and second lines of the above equation are relativistically covariant. Not surprisingly, these are the only terms which remain at zero temperature [9], as can easily be seen if one replaces the sum over n by an integral over p_0 . On the other hand, all of the terms in (8) are invariant under purely spatially-dependent gauge transformations and rotations.

3 Effective Action for Static Modes

Just as at zero temperature, the coefficient of $F_{\mu\nu}^2$ is ultraviolet divergent and needs to be regulated. This can easily be handled by zeta function regularization [9, 10]:

$$\begin{aligned}
[T \sum_{n \neq 0} \int \frac{dt}{\sqrt{t}} e^{-p_0^2 t}]_{\text{reg}} &= T \sum_{n \neq 0} \frac{d}{d\epsilon} \left\{ \frac{\Lambda^{2\epsilon}}{\Gamma(\epsilon)} \int dt t^{\epsilon-1/2} e^{-p_0^2 t} \right\}_{\epsilon \rightarrow 0} \\
&= \frac{1}{\pi} \frac{d}{d\epsilon} \left\{ \left(\frac{\Lambda}{2\pi T} \right)^{2\epsilon} \frac{\Gamma(\epsilon + \frac{1}{2})}{\Gamma(\epsilon)} \zeta(1 + 2\epsilon) \right\}_{\epsilon \rightarrow 0} \\
&= \frac{1}{\sqrt{\pi}} \left\{ \ln\left(\frac{\Lambda}{4T}\right) + \gamma_E \right\}
\end{aligned} \tag{9}$$

where $\gamma_E \sim .577$ is Euler's constant, Λ is the ultraviolet regulator mass, and we have used the fact that we are only summing over nonstatic ($n \neq 0$) fluctuations.

The use of a regulator affects not only the divergent $F_{\mu\nu}^2$ term but also the first term on the second line of (8), even though that term is UV-finite (and would vanish without the regulator). We note that for static fields:

$$\text{tr} \int d^3x A_0 (D_\mu F_{\mu 0}) = -\text{tr} \int d^3x (D_i A_0)^2 = -\text{tr} \int d^3x F_{\mu 0}^2 \tag{10}$$

Using the regulator to perform the frequency sum and t integration, we find:

$$T \sum_{n \neq 0} \frac{d}{d\epsilon} \left\{ \frac{\Lambda^{2\epsilon}}{\Gamma(\epsilon)} \int dt t^{\epsilon-1/2} e^{-p_0^2 t} (1 - 2p_0^2 t) \right\}_{\epsilon \rightarrow 0} = -\frac{1}{\sqrt{\pi}} \tag{11}$$

It is now possible to renormalize in the usual way by defining renormalized quantities in terms of bare (B) quantities via multiplicative constants:

$$A_0^a = Z_E^{-1/2} A_{0,B}^a, \quad A_i^a = Z_M^{-1/2} A_{i,B}^a, \quad g = Z_g^{1/2} g_B \tag{12}$$

Dimensional reduction works best if the Z 's are chosen such that all of the contributions to F_{ij}^2 and F_{i0}^2 which arise from nonstatic loops are cancelled by counterterms [4]. In our case, this can be achieved by choosing:

$$Z_g = Z_M = Z_E - \frac{g^2 N}{12\pi^2} = 1 + \frac{11g^2 N}{24\pi^2} \left[\ln\left(\frac{\Lambda}{4T}\right) + \gamma_E \right] \tag{13}$$

The running coupling $g(T)$ is now uniquely determined and takes the usual form (see for example [4]). I would also like to note that a finite difference between the electric (E) and magnetic (M) wave-function renormalization constants has similarly been found by other authors [4, 5].

For the rest of the terms in eqn. (8), the same result is found whether we sum over n and integrate over t before taking the $\epsilon \rightarrow 0$ limit of the regulator, or if we do things the other way around. To simplify the computation, we therefore choose to take the $\epsilon \rightarrow 0$ limit first, thus effectively removing the regulator. The resulting sums and integrations are:

$$\sum_{n \neq 0} \int dt \sqrt{t} e^{-p_0^2 t} = -\frac{1}{2} \sum_{n \neq 0} \int dt \sqrt{t} (1 - 2p_0^2 t) e^{-p_0^2 t} = \frac{\sqrt{\pi}}{(2\pi T)^3} \zeta(3) \quad (14)$$

$$\sum_{n \neq 0} \int_0^\infty dt \sqrt{t} (3 - 12p_0^2 t + 4p_0^4 t^2) e^{-p_0^2 t} = 0 \quad (15)$$

where $\zeta(x)$ is the Riemann zeta function with $\zeta(3) \sim 1.202$. Furthermore, for static background fields some of the terms can be combined using:

$$\begin{aligned} \text{tr} \int d^3 x \{ (D_0 F_{0\nu})(D_\mu F_{\mu\nu}) \} &= 2ig \text{tr} \int d^3 x \{ A_0 (D_\mu F_{\mu\nu}) F_{0\nu} \} \\ \text{tr} \int d^3 x \{ A_0 (D_\mu^2 D_\nu F_{\nu 0}) \} &= \text{tr} \int d^3 x \{ (D_\mu F_{\mu 0})^2 \} \\ \text{tr} \int d^3 x \{ A_0 (D_0 F_{\mu\nu}) F_{\mu\nu} \} &= 2 \text{tr} \int d^3 x \{ A_0 (D_\mu F_{\mu\nu}) F_{0\nu} + F_{0\mu} F_{\mu\nu} F_{\nu 0} \} . \end{aligned} \quad (16)$$

We now turn to the effective potential. Because we are only interested in static background configurations, the time derivatives vanish from eqn. (6), and we are left with:

$$V'_{\text{eff}} = - \int_0^\infty \frac{dt}{t} \frac{1}{(4\pi t)^{3/2}} T \sum_{n \neq 0} \text{tr} \{ \exp[-(p_0 - gA_0)^2 t] - e^{-p_0^2 t} \} , \quad (17)$$

where the prime on V'_{eff} indicates that we are only integrating over nonstatic fluctuations. Since there are no spatial derivatives in V'_{eff} (or V_{eff}), spatially-dependent

unitary matrices can be used within the trace to transform $A_0(\mathbf{x})$ into a real, diagonal matrix at each point in space. In Appendix D of [11], the $SU(N)$ effective potential for a diagonal background field is calculated to be:

$$\begin{aligned} V_{\text{eff}} &= - \sum_{m=1}^{\infty} \frac{2T^4}{\pi^2 m^4} \text{tr} \{ \cos(mgA_0/T) - 1 \} \\ &= \frac{T^2}{6} \text{tr} \{ (gA_0)^2 (1 - \frac{g|A_0|}{2\pi T})^2 \} , \end{aligned} \quad (18)$$

where $|A_0|$ means to take the absolute value of each (real, diagonal) element in A_0 , and the last line of (18) is only true for fields with $g|A_0| \leq 2\pi T$. To find V'_{eff} , we must subtract the contribution of static ($n = 0$) one loop fluctuations from V_{eff} . Although the full effective potential requires no new renormalization at finite temperature, singling out the static (or non-static) modes does require an additional A_0 mass counterterm [5]. We again choose to use zeta function regularization and find:

$$\begin{aligned} V_{\text{eff}} - V'_{\text{eff}} &= - \frac{T}{(4\pi)^{3/2}} \text{tr} \frac{d}{d\epsilon} \left\{ \frac{\Lambda^{2\epsilon}}{\Gamma(\epsilon)} \int dt t^{\epsilon-5/2} [\exp(-g^2 A_0^2 t) - 1] \right\}_{\epsilon \rightarrow 0} \\ &= - \frac{g^3 T}{6\pi} \text{tr} |A_0|^3 , \end{aligned} \quad (19)$$

where we assumed $\epsilon > 3/2$ to do the integral over t and then continued to $\epsilon = 0$ by implicitly subtracting a (temperature dependent) mass counterterm. (This technique is analogous to continuing from dimension $d < 2$ to $d = 4$ in dimensional mass regularization.) Finally, then, for backgrounds with $g|A_0| \leq 2\pi T$, we get:

$$V'_{\text{eff}} = \frac{g^2 T^2}{6} \text{tr} \{ A_0^2 + \frac{g^2}{(2\pi T)^2} A_0^4 \} \quad (20)$$

This result has been found previously by many authors [4, 5, 6].

4 Conclusion

Putting everything together and including the tree-level action, we find the following expression for the effective action of the static modes of pure gauge $SU(N)$:

$$S_{\text{eff}} = - \frac{1}{T} \text{tr} \int d^3x \left\{ \frac{1}{4N} F_{\mu\nu}^2 + \frac{1}{6} g^2 T^2 A_0^2 + \frac{g^4}{24\pi^2} A_0^4 \right\}$$

$$\begin{aligned}
& - \frac{g^2 \zeta(3)}{64\pi^4 T^2} \left[\frac{1}{90} ig F_{\mu\nu} F_{\nu\rho} F_{\rho\mu} - \frac{19}{60} (D_\mu F_{\mu\nu})^2 - \frac{19}{15} ig F_{0\mu} F_{\mu\nu} F_{\nu 0} \right. \\
& \quad \left. + \frac{1}{30} (D_\mu F_{\mu 0})^2 - \frac{6}{5} ig A_0 (D_\mu F_{\mu\nu}) F_{0\nu} + \frac{11}{6} g^2 A_0^2 F_{\mu\nu}^2 \right] \} \quad (21)
\end{aligned}$$

where g is the running coupling, and the trace is over adjoint representation matrices. Although there are no time derivatives in the above equation, for convenience we use the notation $D_0 = -igA_0$ and $(F_{i0})^{ab} = -if^{abc}(D_i^{cd}A_0^d)$. The top line of this effective action is well known in the literature [4, 5, 6, 7], but the second and third lines contain new terms which may lead to improved agreement of the effective theory with full 4-dimensional QCD at temperatures closer to the phase transition.

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Appendix

To get eqn. (5), we expand the exponential operator of the ghost determinant in powers of covariant derivatives, keeping terms involving up to six covariant derivatives and all terms involving only D_0 . After integrating over d^3p , we are left with a large number of terms which can be regrouped as follows:

$$\begin{aligned}
& 180(4\pi t)^{3/2} \int \frac{d^3p}{(2\pi)^3} \sum_n e^{-p^2 t} \exp[(D^2 + 2iD_\alpha p_\alpha)t] \\
&= \sum_n e^{-p_0^2 t} \{ 180 \exp[(D_0^2 + 2ip_0 D_0)t] \\
&+ 15t^2 [D_\mu, D_\nu]^2 + 30t^2 (1 - 2p_0^2 t) ([D_i, D_0]^2 + \{[D_i, [D_i, D_0]], D_0\}) \\
&+ t^3 (-6\mathcal{O}_1 + \mathcal{O}_2 + 4\mathcal{O}_3 + 3\mathcal{O}_4 + 3\mathcal{O}_5) \\
&+ t^3 (1 - 2p_0^2 t) (-18\mathcal{P}_1 - 6\mathcal{P}_2 - 6\mathcal{P}_3 + 2\mathcal{P}_4 + 2\mathcal{P}_5 + 5\mathcal{P}_6 + 2\mathcal{P}_7 + 4\mathcal{P}_8 \\
&\quad + \frac{3}{2}\mathcal{P}_9 + \frac{3}{2}\mathcal{P}_{10} + 6\mathcal{P}_{11} + 6\mathcal{P}_{12} - 6\mathcal{P}_{13} - 6\mathcal{Q}_1 + 6\mathcal{Q}_2 + \frac{3}{2}\mathcal{Q}_3 - \frac{3}{2}\mathcal{Q}_4 \\
&\quad + \frac{9}{2}\mathcal{Q}_5 - \frac{9}{2}\mathcal{Q}_6 + 15\mathcal{Q}_7 + 3\mathcal{Q}_8 - 12\mathcal{Q}_9 + 12\mathcal{Q}_{10} + 18\mathcal{Q}_{11} - 6\mathcal{Q}_{12} + 3\mathcal{Q}_{14}) \\
&+ 2t^3 (3 - 12p_0^2 t + 4p_0^4 t^2) (-\mathcal{P}_{13} - 3\mathcal{Q}_9 + 3\mathcal{Q}_{10} + 6\mathcal{Q}_{11} + 3\mathcal{Q}_{12} + 5\mathcal{Q}_{13} + 2\mathcal{Q}_{14}) \}
\end{aligned} \tag{A}$$

where curly brackets indicate anticommutators, i and j are used for spatial indices only, and the operators which arise in the sixth order of the expansion are given below.

The \mathcal{O} operators were originally presented in in the context of a zero temperature covariant derivative expansion [9]. These completely gauge invariant and relativistically covariant commutators are given by:

$$\begin{aligned}
\mathcal{O}_1 &= [D_\mu, D_\nu][D_\nu, D_\rho][D_\rho, D_\mu] \\
\mathcal{O}_2 &= [D_\mu, [D_\mu, D_\rho]][D_\nu, [D_\nu, D_\rho]] & \mathcal{O}_3 &= [D_\mu, [D_\nu, D_\rho]][D_\mu, [D_\nu, D_\rho]] \\
\mathcal{O}_4 &= [D_\mu, [D_\mu, [D_\nu, D_\rho]]][D_\nu, D_\rho] & \mathcal{O}_5 &= [D_\nu, D_\rho][D_\mu, [D_\mu, [D_\nu, D_\rho]]]
\end{aligned}$$

Although the \mathcal{P} operators are not relativistically covariant, they are completely gauge invariant:

$$\mathcal{P}_1 = [D_0, D_i][D_i, D_j][D_j, D_0] \quad \mathcal{P}_2 = [D_i, D_0][D_0, D_j][D_j, D_i]$$

$$\begin{aligned}
\mathcal{P}_3 &= [D_i, D_j][D_j, D_0][D_0, D_i] & \mathcal{P}_4 &= [D_0, [D_0, D_i]][D_j, [D_j, D_i]] \\
\mathcal{P}_5 &= [D_j, [D_j, D_i]][D_0, [D_0, D_i]] & \mathcal{P}_6 &= [D_i, [D_i, D_0]][D_j, [D_j, D_0]] \\
\mathcal{P}_7 &= [D_0, [D_i, D_j]][D_0, [D_i, D_j]] & \mathcal{P}_8 &= [D_i, [D_j, D_0]][D_i, [D_j, D_0]] \\
\mathcal{P}_9 &= [D_0, [D_0, [D_i, D_j]]][D_i, D_j] & \mathcal{P}_{10} &= [D_i, D_j][D_0, [D_0, [D_i, D_j]]] \\
\mathcal{P}_{11} &= [D_i, [D_i, [D_j, D_0]]][D_j, D_0] & \mathcal{P}_{12} &= [D_j, D_0][D_i, [D_i, [D_j, D_0]]] \\
\mathcal{P}_{13} &= [D_0, [D_0, D_i]][D_0, [D_0, D_i]]
\end{aligned}$$

The \mathcal{Q} operators are only invariant under spatially dependent gauge transformations:

$$\begin{aligned}
\mathcal{Q}_1 &= D_0[D_i, [D_i, D_j]][D_0, D_j] & \mathcal{Q}_2 &= [D_0, D_j][D_i, [D_i, D_j]]D_0 \\
\mathcal{Q}_3 &= D_0[D_0, [D_i, D_j]][D_i, D_j] & \mathcal{Q}_4 &= [D_i, D_j][D_0, [D_i, D_j]]D_0 \\
\mathcal{Q}_5 &= D_0[D_i, D_j][D_0, [D_i, D_j]] & \mathcal{Q}_6 &= [D_0, [D_i, D_j]][D_i, D_j]D_0 \\
\mathcal{Q}_7 &= D_0[D_i, D_j]^2 D_0 & \mathcal{Q}_8 &= \{D_0, [D_i, [D_i, [D_j, [D_j, D_0]]]]\} \\
\mathcal{Q}_9 &= D_0[D_0, [D_0, D_i]][D_0, D_i] & \mathcal{Q}_{10} &= [D_0, D_i][D_0, [D_0, D_i]]D_0 \\
\mathcal{Q}_{11} &= \{D_0^2, [D_0, D_i]^2\} & \mathcal{Q}_{12} &= D_0[D_0, D_i]^2 D_0 \\
\mathcal{Q}_{13} &= D_0\{D_0, [D_i, [D_i, D_0]]\}D_0 & \mathcal{Q}_{14} &= \{D_0[D_0, [D_0, [D_i, [D_i, D_0]]]]\}
\end{aligned}$$

When traced and integrated over spacetime, all of the \mathcal{O} operators can be expressed in terms of \mathcal{O}_1 and \mathcal{O}_2 :

$$\text{tr} \int d^4x \mathcal{O}_3 = -\text{tr} \int d^4x \mathcal{O}_4 = -\text{tr} \int d^4x \mathcal{O}_5 = \text{tr} \int d^4x (-4\mathcal{O}_1 + 2\mathcal{O}_2)$$

Similarly, we can express all of the \mathcal{P} 's in terms of \mathcal{P}_1 , \mathcal{P}_4 , \mathcal{P}_6 , and \mathcal{P}_{13} :

$$\text{tr} \int d^4x \mathcal{P}_2 = \text{tr} \int d^4x \mathcal{P}_3 = \text{tr} \int d^4x \mathcal{P}_1$$

$$\text{tr} \int d^4x \mathcal{P}_5 = \text{tr} \int d^4x \mathcal{P}_4$$

$$\text{tr} \int d^4x \mathcal{P}_7 = -\text{tr} \int d^4x \mathcal{P}_9 = -\text{tr} \int d^4x \mathcal{P}_{10} = \text{tr} \int d^4x (-4\mathcal{P}_1 + 2\mathcal{P}_4)$$

$$\text{tr} \int d^4x \mathcal{P}_8 = -\text{tr} \int d^4x \mathcal{P}_{11} = -\text{tr} \int d^4x \mathcal{P}_{12} = \text{tr} \int d^4x (-4\mathcal{P}_1 + \mathcal{P}_4 + \mathcal{P}_6),$$

and all of the \mathcal{Q} 's in terms of \mathcal{Q}_1 , \mathcal{Q}_3 , \mathcal{Q}_7 , \mathcal{Q}_8 , \mathcal{Q}_9 , \mathcal{Q}_{12} , and \mathcal{Q}_{13} , and \mathcal{Q}_{14} :

$$\text{tr} \int d^4x \mathcal{Q}_2 \mathbf{1} = -\text{tr} \int d^4x \mathcal{Q}_1 \mathbf{1}$$

$$\text{tr} \int d^4x \mathcal{Q}_4 \mathbf{1} = \text{tr} \int d^4x \mathcal{Q}_5 \mathbf{1} = -\text{tr} \int d^4x \mathcal{Q}_6 \mathbf{1} = -\text{tr} \int d^4x \mathcal{Q}_3 \mathbf{1}$$

$$\text{tr} \int d^4x \mathcal{Q}_{10} \mathbf{1} = -\text{tr} \int d^4x \mathcal{Q}_9 \mathbf{1} \quad \text{tr} \int d^4x \mathcal{Q}_{11} \mathbf{1} = 2\text{tr} \int d^4x \mathcal{Q}_{12} \mathbf{1} .$$

When these substitutions are made in (A), the result is eqn. (5)

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